CONCORDANCE MEASURES AND SECOND ORDER STOCHASTIC DOMINANCE – PORTFOLIO EFFICIENCY ANALYSIS

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Introduction

The portfolio selection problem is one of the most important issues of financial risk management. In order to determine the optimal composition for a particular portfolio it is crucial to estimate the dependency among the evolution of particular risk factors, i.e., the joint distribution of log-returns of particular assets. However, in order to formulate the joint distribution, there is a need for a suitable measure of dependency. A standard assumption is that the (joint) distribution of large portfolios is multivariate normal and that the dependency can be described well by a linear measure of correlation (Pearson coefficient of correlation). Unfortunately, from real applications it is clear that the Pearson correlation is not sufficiently robust to describe the dependency of market returns (see e.g. [22]).

Among more advanced candidates for a suitable dependency measure we can classify the well known concordance measures such as Kendall's tau or Spearman's rho. Minimizing these alternative measures of portfolio's risk one can obtain several distinct "optimal" portfolios. The question is how to compare these portfolios among each other. The easiest way is based on comparisons of portfolios' means. Since seminal work of Markowitz [15] has been introduced, see also [16], the portfolios has been described by mean and variance. Besides that, however, some other measures of risk were proposed instead of variance. Anyway, there is no general agreement in the question of the "true" risk measure. Therefore, the alternative ways of portfolio comparisons were developed.

Stochastic dominance approach is one of the most popular one. Stochastic dominance was introduced independently in [6], [7], [23] and [26]. The definition of second-order stochastic dominance (SSD) relation uses comparisons of either twice cumulative distribution functions, or expected utilities (see for example [2], [3] or [12]). Alternatively, one can define SSD relation using cumulative quantile functions or conditional value at risk (see [18] or [8]).

Similarly to the well-known mean-variance criterion, the second-order stochastic dominance relation can be used in portfolio efficiency analysis as well. A given portfolio is called SSD efficient if there exists no other portfolio preferred by all risk-averse and risk-neutral decision makers (see for example [24], [11] or [8]). To test SSD efficiency of a given portfolio relative to all portfolios created from a set of assets [21], [11] and [8] proposed several linear programming algorithms. While the Post test is based on representative utility functions and strict dominance criterion, in order to search for a utility function satisfying optimality criterion, the Kuosmanen and the Kopa-Chovanec test focus on the identification of a dominating portfolio. For SSD inefficient portfolios, several SSD portfolio inefficiency measures were introduced in [24], [11] and [8]. These measures are based on a "distance" between a tested portfolio and some other portfolio identified by a SSD portfolio efficiency test. For SSD efficient portfolio, [10] developed a measure of SSD efficiency as a measure of stability with respect to changes in probability distribution of returns. In all SSD portfolio efficiency tests the scenario approach is assumed, that is, the returns of assets are modeled by discrete distribution with equiprobable scenarios. This assumption is not very restrictive, because every discrete multivariate distribution with rational probabilities can be represented by equiprobable scenarios where some of the scenarios may be repeated. Besides second-order stochastic dominance, one can use first-order stochastic dominance in portfolio efficiency analysis. See [11] and [9] for details.

In this paper we try to examine the efficiency of selected portfolios by terms of second order stochastic dominance because we assume risk averse decision makers. Our main idea is that there might by some impact of (i) alternative dependency measures and/or (ii) shortselling constrains on the efficiency of a min-var portfolio. Therefore, we identify several distinct min-var portfolios on the basis of alternative concordance matrix as defined in [19]. We also consider two types of restrictions on short sales (Black model and Markowitz model), three measures of dependency/concordance (Pearson, Spearman and Kendall) and two data sets (year 2007 and year 2008), so that we get 12 distinct portfolios in total. We apply the Kuosmanen SSD efficiency test to these portfolios in order to analyze their SSD efficiency. More particularly, the SSD efficiency/inefficiency measure is evaluated for each portfolio and the impact of short sales restriction and choice of measure of concordance on the SSD efficiency/ /inefficiency of min-var portfolios is analyzed. Special attention is paid to the comparison of the pre-crises with the starting-crises results.

We proceed as follows. First, in Section 1, we summarize the basic theoretical concepts of concordance measures and portfolio selection problem. Next, in Section 2, stochastic dominance approach with a special focus on portfolio efficiency with respect to second-order stochastic dominance (SSD) criterion is presented. Moreover, three linear programming tests are briefly recalled. In Section 3, we continue with a numerical study: We identify 12 min-var portfolios first and then we test their SSD efficiency. Finally, we calculate SSD efficiency/inefficiency measures of these portfolios to be able to compare their SSD performance. In the last section the most important conclusions and remarks are stated.

1. Concordance Measures and Portfolio Selection

Let us consider a random vector $r = (r_1, r_2, ..., r_n)'$ of returns of n assets with discrete probability distribution described by T equiprobable scenarios. The returns of the assets for the various scenarios are given by

$$X = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^T \end{pmatrix}$$

where $x^t = (x_1^t, x_2^t, ..., x_n^t)$ is the t-th row of matrix X. We will use $w = (w_1, w_2, ..., w_n)$ for the vector of portfolio weights. Throughout the paper, we will consider two special sets of portfolio weights:

$$W_M = \left\{ w \in \mathrm{R}^n : \sum_{i=1}^{n} w_i = 1,\ w_i \geq 0,\ i = 1, 2, \ldots, n \right\}$$

$$W_B = \left\{ w \in \mathrm{R}^n : \sum_{i=1}^{n} w_i = 1,\ w_i \geq -1,\ i = 1, 2, \ldots, n \right\}.$$

Besides that, we use the following notation: expected returns $m = (\mu_1, \mu_2, ..., \mu_n)'$, standard deviations of returns $s = (\sigma_1, \sigma_2, ..., \sigma_n)'$, and correlation matrix $R = [\rho_{i,j}]$, i.e. it consists of all combinations of Pearson linear coefficient of correlation ρ_{ij}, where $i, j = 1, ... n$.

Following the standard portfolio selection problem of Markowitz [15] no riskless investment is allowed and only the mean return and the risk measure of standard deviation matter, mainly since the Gaussian distribution of price returns is assumed. In such a setting, the efficient frontier of portfolios, i.e., the only combination of particular assets that should be considered for risky investments, is bounded by minimal variance portfolio, Π_A, from the left and maximal return portfolio, Π_B, from the right. We can obtain them as follows.

Task 1 Minimal variance portfolio, Π_A

$$\begin{array}{llrcl} \mathrm{var}(\Pi) \longrightarrow \min. & with & \mathrm{var}(\Pi) & = & w'\Sigma w \\ & & \Sigma & = & [\sigma_i \sigma_j \rho_{ij}] \\ & s.t. & w'1 & = & 1. \\ & & w & \geq & 0. \end{array}$$

Task 2 Maximal return portfolio, Π_B

$$\begin{array}{llrcl} \mathrm{var}(\Pi) \longrightarrow \max. & with & \mu(\Pi) & = & w'm \\ & s.t. & w'1 & = & 1. \\ & & w & \geq & 0. \end{array}$$

Alternatively, Task 1 (Task 2) can be solved subject to $w_i \geq -1$, $i = 1, 2, ..., n$, i.e., short positions in any of the assets are allowed with no restriction on long positions (*Black Model* [1]).

The optimal portfolio under both models depends on preferences of a particular investor. It can be detected on the basis of a given utility function, a performance measure (Sharpe ratio, information ratio, etc.), a risk measure (VaR, CVaR) or their combinations.

Obviously, the composition of any portfolio, except the maximal return one, will depend on the correlation matrix. The elements of the correlation matrix *R*, i.e., a crucial factor to determine optimal weights for Π_A, describe the linear dependency among two variables. The main drawback is that it can be zero even if the variables are dependent and it does not take into account tail dependency. It follows that the correlation is suitable mainly for problems with elliptically distributed random variables. Since the assumption about the Gaussianity of financial returns is unjustifiable – this observation goes back to early 60's, see e.g. [13], [14] or [4] – there is a need for alternative measures, which should allow us to obtain better performance, diversification or both.

A general family of measures that is not restricted to the case of linear dependency consists of *concordance measures*. A measure of concordance is any measure that is normalized to the interval [–1,1] and pays attention not only to the dependency but also to the co-monotonicity and anti-monotonicity. For more details on all properties of concordance measures and their proofs see e.g. [17].

Following [17], two random variables (X,Y) with independent replications (x_1,y_1) and (x_2,y_2) are concordant if $x_1 < x_2$ $(x_1 > x_2)$ implies $y_1 < y_2$ $(y_1 > y_2)$. Similarly, the two variables are discordant if $x_1 > x_2$ $(x_1 > x_2)$ implies $y_1 < y_2$ $(y_1 < y_2)$. The concordance measures are easily definable by copula functions, since they rely only on the "joint" features, having no relations to the marginal characteristics. There are two popular measures of concordance – Kendall's tau and Spearman's rho, which can be accompanied by the following measures of association: Gini's gamma and Blomqvist's beta.

The first measure of concordance in mind is Kendall's tau, τ_K, since it is defined as the probability of concordance reduced by the probability of discordance:

$$\tau_K(X,Y) = \mathrm{P}((x_1 - x_2)(y_1 - y_2) > 0) - \mathrm{P}((x_1 - x_2)(y_1 - y_2) < 0). \quad (1)$$

with the following simplification for continuous variables:

$$\tau_K(X,Y) = 2\mathrm{P}((x_1 - x_2)(y_1 - y_2) > 0) - 1. \quad (2)$$

For n observations it can be estimated on the basis of observations of concordance *(c)* and disconcordance *(d)* as follows:

$$\tau_K(X,Y) = \frac{c - d}{c + d}. \quad (3)$$

In order to define the second popular measure of concordance, Spearman's rho, ρ_S, the third realization of both random variables, (x_3,y_3), should be considered:

$$\rho_S(X,Y) = 3(\mathrm{P}((x_1 - x_2)(y_1 - y_3) > 0) - \mathrm{P}((x_1 - x_2)(y_1 - y_3) < 0)). \quad (4)$$

It means that the Spearman's rho is given as the probability of concordance reduced by the probability of discordance, in contrast to the Kendall's tau, for the pairs (x_1,y_1) and (x_2,y_3). It also implies that this measure is very similar to the linear correlation coefficient, except the fact, that it measures the dependency among marginal distribution functions:

$$\rho_S(X,Y) = \frac{\mathrm{cov}(F_X(x), F_Y(y))}{\sqrt{\mathrm{var}(F_X(x))\,\mathrm{var}(F_Y(y))}}. \quad (5)$$

It follows, that it can be regarded as the correlation of copula functions. The proof that all the measures introduced in this section are really measures of concordance can be found e.g. in [17].

Thus, being equipped with formulas to calculate (estimate) alternative dependency measures, we can replace the elements of the covariance matrix S from Task 1:

$$\mathrm{cov}(X,Y) = \sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}\rho(X,Y)$$

by the elements of a concordance matrix e.g. by terms of Spearman's rho (4):

$$\mathrm{cov}_S(X,Y) - \sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}\rho_S(X,Y).$$

or Kendall's tau (1):

$$\mathrm{cov}_K(X,Y) \quad \sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}\rho_K(X,Y).$$

2. Second Order Stochastic Dominance and Portfolio Efficiency

Stochastic dominance relation allows comparison of two portfolios via comparison of their random returns. Let $F_{r'w}(x)$ denote the cumulative probability distribution function of returns of portfolio with weights w. Since each portfolio is uniquely given by its weight vector we will shortly denote this portfolio by w, too. The twice cumulative probability distribution function of returns of portfolio w is given by:

$$F^{(2)}_{r'w}(t) = \int_{-\infty}^{t} F_{r'w}(x)dx$$

and we say that portfolio v dominates portfolio w by second-order stochastic dominance ($r'v >_{SSD} r'w$) if and only if

$$F^{(2)}_{r'v}(t) \leq F^{(2)}_{r'w}(t) \quad \forall t \subset \mathrm{R}$$

with strict inequality for at least one $t \in$ R This relation is sometimes called strict second-order stochastic dominance because the strict inequality for at least one $t \in$ R is required, see [11] for more details. Alternatively, one may use several different ways of defining the second-order stochastic dominance (SSD) relation:

- $r'v >_{SSD} r'w$ if and only if $Eu(r'v) \geq Eu(r'w)$ for all concave utility functions u provided the expected values above are finite and strict inequality is fulfilled for at least some concave utility function, see for example [11].
- $r'v >_{SSD} r'w$ if and only if $F^{(-2)}_{r'v}(p) \geq F^{(-2)}_{r'w}(p)$ for all $p \in \langle 0,1\rangle$ with strict inequality for at least some p where second quantile functions $F^{(-2)}_{r'v}$, $F^{(-2)}_{r'w}$, are convex conjugate functions of $F^{(2)}_{r'v}$ and $F^{(2)}_{r'w}$, respectively, in the sense of Fenchel duality, see [18].
- $r'v >_{SSD} r'w$ if and only if $CVaR_\alpha(-r'v) \leq$ $\leq CVaR\alpha(-r'w)$ for all $\alpha \in \langle 0,1\rangle$ where conditional value at risk (CVaR) of portfolio w can be defined via optimization problem:

$$\begin{aligned} \mathrm{CVaR}_\alpha(-r'w) &= \min_{a,z_t} a + \frac{1}{(1-\alpha)T}\sum_{t=1}^{T} z_t \qquad (6)\\ \text{s.t. } z_t &\geq -x^t w - a \\ z_t &\geq 0. \end{aligned}$$

and for portfolio v as:

$$\begin{aligned} \mathrm{CVaR}_\alpha(-r'v) &= \min_{a,z_t} a + \frac{1}{(1-\alpha)T}\sum_{t=1}^{T} z_t \\ \text{s.t. } z_t &\geq -x^t v - a \\ z_t &\geq 0. \end{aligned}$$

See [11], [25] and [20] for more details.

We say that portfolio $w \in W_M$ is SSD inefficient with respect to W_M if and only if there exists portfolio $v \in W_M$ such that $r'v >_{SSD} r'w$. Otherwise, portfolio w is SSD efficient with respect to W_M. By analogy, portfolio $w \in W_B$ is SSD inefficient with respect to W_B if and only if there exists portfolio $v \in W_B$ such that $r'v >_{SSD} r'w$. This definition classifies portfolio $w \in W_M$ or $w \in W_B$ as SSD efficient if and only if no other portfolio from W_M or W_B is better (in the sense of the SSD relation) for all risk averse and risk neutral decision makers.

Since the decision maker may form infinitely many portfolios, the criteria for pairwise comparisons have only limited use in portfolio efficiency testing. To test whether a given portfolio w is SSD efficient, three linear programming tests were developed. We formulate the tests for SSD efficiency with respect to W_M. However, one can easily rewrite it for W_B.

2.1 The Post SSD Portfolio Efficiency Test

The first SSD portfolio efficiency test was introduced in [21]. Before testing SSD efficiency of portfolio w, one must order the rows of scenario matrix X in such a way that $x^1w \leq x^2w \leq$ $\leq \ldots x^Tw$. The test requires solution of the following linear program:

$$\begin{aligned} \theta^*(w) = \min_{\theta,\beta_t} \theta & \\ \text{s.t. } \sum_{t=1}^{T} \beta_t(x^t w - x^t_i) + T\theta &\geq 0 \quad i = 1,2,\ldots,n \\ \beta_t - \beta_{t+1} &\geq 0 \quad t = 1,2,\ldots,T-1 \\ \beta_t &\geq 0 \quad t = 1,2,\ldots,T-1 \\ \beta_T &\geq 1 \end{aligned}$$

If $\theta^*(w) > 0$ then portfolio w is SSD inefficient with respect to W_M.

If some ties in elements of X occur, then the constraints should be modified. See [21] for more details. Anyway, this criterion failed to detect SSD inefficiency of portfolio w when comparing portfolios with identical means. It does not detect the presence of SSD dominating portfolio if mean of its returns equals to mean return of portfolio w. Therefore, the Post test is only a necessary criterion for SSD portfolio efficiency with respect to W_M. This is the reason why the other two tests were developed.

2.2 The Kuosmanen SSD Portfolio Efficiency Test

The SSD efficiency test proposed in [11] is based on second quantile criterion. The second quantile function of portfolio w can be rewritten in terms of cumulative returns under scenario approach assumption, that is:

$$\begin{aligned} F_{r'w}^{(-2)}\left(\frac{k}{T}\right) &= \frac{1}{T}\sum_{t=1}^{k}(Xw)^{[t]} \quad k = 1,2,\dots,T \\ F_{r'w}^{(-2)}(0) &= 0 \end{aligned}$$

where $(Xw)^{[t]}$ denotes the t-th smallest return of portfolio w, that is, one has: $(Xw)^{[1]} \le (Xw)^{[2]} \le \dots \le (Xw)^{[T]}$. Combining it with majorization theorem (principle), see Hardy et al. [7, Thm 46]:

$\sum_{t=1}^{k}(Xw)^{[t]} \le \sum_{t=1}^{k}(Xv)^{[t]}$, $k = 1, 2, \dots, T-1$ and
$\sum_{t=1}^{T}(Xw)^{[t]} = \sum_{t=1}^{T}(Xv)^{[t]}$

$\Leftrightarrow$ $\exists$ double stochastic matrix P such that $PXw = Xv$ we have another criterion for SSD relation:

$r'v >_{SSD} r'w$ if and only if there exists a double stochastic matrix $P = \{p_{ij}\}_{i,j=1}^{T}$ such that ($PXw \le Xv$ and $1'PXw < 1'Xv$) or ($PXw = Xv$ and $\sum_{i=1}^{T} p_{ii} < 1$) where $1' = (1, 1, \dots, 1)$. See [11] and [7, Thm 46] for more details.

Using this criterion, [11] proposed the SSD efficiency test consisted of solving two linear programs, in order to identify a dominating portfolio (if it exists) which is already SSD efficient. Let

$$\begin{aligned} \varphi^*(w) = \min_{P,v} \sum_{t=1}^{T} &(r'v - r'w) \\ \text{s.t.} \quad PXw &\le Xv \\ \sum_{j=1}^{T} p_{ij} = 1, \quad \sum_{i=1}^{T} p_{ij} &= 1, \quad p_{ij} \ge 0 \quad i,j = 1,2,\dots,T \\ v &\in W_M \end{aligned} \tag{7}$$

and

$$\begin{aligned} \varphi^{**}(w) = \min_{P,v,S^+,S^-} &\sum_{j=1}^{T}\sum_{i=1}^{T}(s_{ij}^+ + s_{ij}^-) \\ \text{s.t.} \quad Xv &= PXw \\ s_{ij}^+ - s_{ij}^- &= p_{ij} - \frac{1}{2} \quad i,j = 1,2,\dots,T \\ s_{ij}^+, s_{ij}^-, p_{ij} &\ge 0 \quad i,j = 1,2,\dots,T \\ \sum_{j=1}^{T} p_{ij} = 1, \quad \sum_{i=1}^{T} p_{ij} &= 1 \quad i,j = 1,2,\dots,T \\ v &\in W_M \end{aligned} \tag{8}$$

where $S^+ = \{s_{ij}^+\}_{i,j=1}^{T}$, $S^- = \{s_{ij}^-\}_{i,j=1}^{T}$, and $P = \{p_{ij}\}_{i,j=1}^{T}$. Let ε_k denote the number of k-way ties in Xw. Then portfolio w is SSD efficient with respect to W_M if and only if

$$\varphi^*(w) = 0 \ \wedge \ \varphi^{**}(w) = \frac{T^2}{2} - \sum_{k=1}^{T} k\varepsilon_k$$

If $\varphi^*(w) > 0$ then problem (9) need not to be solved, because portfolio w is SSD inefficient with respect to W_M and the optimal solution v^* is an SSD dominating portfolio which is already SSD efficient with respect to W_M, see [11] for more details.

2.3 The Kopa-Chovanec SSD Portfolio Efficiency Test

In this section we present the SSD portfolio efficiency linear programming test in the form of a necessary and sufficient condition derived in [8]. This test is based on CVaR comparison criterion. Under scenario assumption, the criterion can be reduced to T inequalities:

$r'v >_{SSD} r'w$ if and only if

$$CVaR_{\frac{k-1}{T}}(-r'v) \le CVaR_{\frac{k-1}{T}}(-r'w)$$

for all $k = 1, 2, \dots, T$ with at least one strict inequality and portfolio w is SSD efficient with respect to W_M if and only if the optimal value of the following problem:

$$\begin{aligned} \max_{v,S_k} \sum_{k=1}^{T} &S_k \\ \text{s.t.} \quad CVaR_{\frac{k-1}{T}}(-r'w) - CVaR_{\frac{k-1}{T}}(-r'v) &\ge S_k \\ S_k &\ge 0 \\ v &\in W_M \end{aligned} \tag{9}$$

is strictly positive. Applying (6), [8] derived from (9) the linear programming SSD efficiency test:

Let

$$\begin{aligned} S^*(w) = \max_{S_k, v, b_k, z_t^k} &\sum_{k=1}^{T} S_k \\ \text{s.t.} \quad CVaR_{\frac{k-1}{T}}(-r'w) - b_k - \frac{1}{(T-k+1)}\sum_{t=1}^{T} z_t^k &\ge S_k \quad k = 1,2,\dots,T \\ z_t^k &\ge 0 \quad t,k = 1,2,\dots,T \\ z_t^k + x^t v + b_k &\ge 0 \quad t,k = 1,2,\dots,T \\ S_k &\ge 0 \quad k = 1,2,\dots,T \\ v &\in W_M \end{aligned} \tag{10}$$

If $S^*(w) > 0$ then portfolio w is SSD inefficient with respect to W_M and $r'v^* >_{SSD} r'w$. Otherwise $S^*(w) = 0$, and portfolio w is SSD efficient with respect to W_M.

If a given portfolio is SSD inefficient with respect to W_M then the test identifies a dominating portfolio which is SSD efficient with respect to W_M. Comparing to the Kuosmanen test, this test makes use of asymptotically (for large number of scenarios) six-times smaller linear program than the second problem of the Kuosmanen test (8). On the other hand, for an SSD inefficient portfolio with respect to W_M, the Kuosmanen necessary test (7) identifies a dominating portfolio by solving asymptotically two-times smaller linear program than the Kopa-Chovanec test. If portfolio w is SSD inefficient with respect to W_M then the Kuosmanen test identifies a SSD dominating portfolio with the highest mean return, while the Kopa-Chovanec test chooses a SSD dominating portfolio with the minimal risk measured by the average CVaR:

$$\frac{1}{T}\sum_{k=1}^{T} CVaR_{\frac{k}{T}}\,(r'v).$$

More details about the Kopa-Chovanec SSD portfolio efficiency test and a comparison of all three tests can be found in [8].

3. Empirical Study

Let us consider daily quotes of FX rates for EUR, GBP, HUF, PLN, SKK, and USD, each with respect to CZK (www.cnb.cz). We pick up daily log-returns over 2007 and 2008, i.e., we get approximately 6 x 250 log-returns for both time series. In this way, we can compare the results for pre-crisis period (year 2007) and starting crisis period (year 2008).

The first task is to determine the optimal weights of particular currencies for min-var portfolios following either the approach of Markowitz (no short selling) or Black (short selling up to the initial investment is allowed) on the basis of three distinct dependence/ /concordance matrices Σ. That is, the min-var optimal portfolios are obtained as the optimal solutions of the following quadratic program:

$$\text{var}\,(\Pi) \Rightarrow \min, \text{ with } \text{var}\,(\Pi) = w'\Sigma w$$

$$\Sigma = [\sigma_i \sigma_j \sigma_{dij}] \text{ s.t. } w \in W_M$$

for the Markowitz model case and

$$\text{var}\,(\Pi) \Rightarrow \min, \text{ with } \text{var}\,(\Pi) = w'\Sigma w$$

$$\Sigma = [\sigma_i \sigma_j \sigma_{dij}] \text{ s.t. } w \in W_B$$

for the Black model case, where d_{ij} is an element of either Pearson, Spearman, or Kendall matrix of dependence/concordance. See Table 1 and 2 for review of all portfolios we deal with. In the same table we provide for each portfolio the 2007 mean return, classic standard deviation and concordance measure C defined as

$$C = \sqrt{w'\Sigma w}.$$

Similarly, the same results based on 2008 data are provided in Table 2.

Tab. 1: Denotation of particular portfolios and their characteristics, 2007

Portfolio	Correlation *(R)*	Short selling	Mean return (%)	Stand.dev.	Measure *C*
Π_M1	Pearson	No	–0.00711	0.00547	0.00547
Π_B1	Pearson	Yes	0.00711	0.00539	0.00539
Π_M2	Spearman	No	0.00009	0.00549	0.00562
Π_B2	Spearman	Yes	0.00234	0.00547	0.00562
Π_M3	Kendall	No	–0.00182	0.00556	0.00518
Π_B3	Kendall	Yes	–0.00182	0.00556	0.00518

Source: authors' calculation

Tab. 2: Denotation of particular portfolios and their characteristics, 2008

Portfolio	Correlation *(R)*	Short selling	Mean return (%)	Stand.dev.	Measure *C*
Π_M1	Pearson	No	–0.01404	0.00260	0.00260
Π_B1	Pearson	Yes	–0.00869	0.00255	0.00255
Π_M2	Spearman	No	–0.01478	0.00260	0.00254
Π_B2	Spearman	Yes	–0.00958	0.00255	0.00250
Π_M3	Kendall	No	–0.01675	0.00262	0.00235
Π_B3	Kendall	Yes	–0.01625	0.00261	0.00235

Source: authors' calculation

Following these two tables we can see that all six min-var portfolios for 2008 have smaller mean return and smaller standard deviation than corresponding portfolios for 2007 data. Perhaps surprisingly, the values of measures of concordance are very close to corresponding standard deviations. We will proceed with SSD portfolio efficiency testing of these 12 portfolios. It is well known, that portfolios with minimal standard deviation generally need not to be SSD efficient. To see it, consider the following simple example. See [11] for more details.

Example 1

Let $X = \begin{pmatrix} 1 & 10 \\ 1 & 2 \end{pmatrix}$, that is, we consider only two assets and two equiprobable scenarios for their returns. Let portfolio w = (1, 0) and portfolio v = (1, 0). There is no doubt that w is the portfolio with minimal standard deviation. It is easy to check that $r'v >_{SSD} r'w$ because every non-satiated, risk averse or risk neutral decision maker prefers portfolio v to portfolio w. Hence portfolio w is SSD inefficient.

If we use CVaR as a measure of risk one can show that if a portfolio with minimal CVaR is uniquely determined then it is SSD efficient. This property follows from SSD criterion expressed in terms of CVaR. The aim of this paper is to analyze the relationship between portfolios with minimal risk and SSD efficient portfolios when three considered concordance measures are used as measures of dependency.

Since 12 considered min-var portfolios are constructed as portfolios with minimal risk we expect that they have relatively small mean returns. Therefore we choose the Kuosmanen test for SSD portfolio efficiency testing. If the tested portfolio is SSD inefficient, the Kousmanen test gives us information about SSD dominating portfolio with the highest mean return and the SSD inefficiency measure identifies the maximal possible improvement (in terms of mean returns) that can be done by moving from a min-var portfolio to better ones (in sense of SSD relation). The results of the Kuosmanen test for considered portfolios are summarized in Table 3 and Table 4.

Tab. 3: SSD efficiency results of min-var portfolios, 2007

Portfolio	Correlation (*R*)	Short selling	SSD efficiency	Mean return (%)	Mean return of SSD dominating portfolio (%)	Measure *C*	Measure *C* of SSD dominating portfolio
Π_M1	Pearson	No	No	–0.00711	–0.00700	0.00547	0.00547
Π_B1	Pearson	Yes	No	0.00711	0.02343	0.00539	0.00547
Π_M2	Spearman	No	No	0.00009	0.00017	0.00562	0.00562
Π_B2	Spearman	Yes	No	0.00234	0.02435	0.00562	0.00579
Π_M3	Kendall	No	No	–0.00182	0.01063	0.00518	0.00529
Π_B3	Kendall	Yes	No	–0.00182	0.03741	0.00518	0.00591

Source: authors' calculation

Tab. 4: SSD efficiency results of min-var portfolios, 2008

Portfolio	Correlation (*R*)	Short selling	SSD efficiency	Mean return (%)	Mean return of SSD dominating portfolio (%)	Measure *C*	Measure *C* of SSD dominating portfolio
Π_M1	Pearson	No	No	–0.01404	–0.01285	0.00260	0.00260
Π_B1	Pearson	Yes	No	–0.00869	–0.00123	0.00255	0.00260
Π_M2	Spearman	No	No	–0.01478	–0.01437	0.00254	0.00254
Π_B2	Spearman	Yes	No	–0.00958	–0.00245	0.00250	0.00255
Π_M3	Kendall	No	No	–0.01675	–0.01090	0.00235	0.00242
Π_B3	Kendall	Yes	No	–0.01625	0.00690	0.00235	0.00291

Source: authors' calculation

Table 3 and Table 4 show us that all min-var portfolios were classified as SSD inefficient and for every min-var portfolio exists some SSD dominating portfolio with higher mean return. Comparing the mean returns and measures of concordance of SSD dominating portfolios for 2007 data with that for 2008 data, we can conclude that all SSD dominating portfolios for 2007 data have higher mean return and higher measure of concordance. The same property was observed for min-var portfolios.

Finally we compare the differences of mean returns and measures of concordance between min-var portfolios and their SSD dominating portfolios. Firstly, we evaluate the absolute differences of mean returns (it is equal to the SSD inefficiency measure) and absolute differences of measures of concordance. To be able to compare the differences between each other we compute the relative differences of mean returns and relative differences of measures of concordance for all 12 portfolios. The results are summarized in Table 5 and Table 6.

Tab. 5: Differences between min-var portfolios and their SSD dominating portfolios, 2007

Portfolio	Correlation *(R)*	Short selling	Absolute difference of mean returns	Absolute difference of measure *C*	Relative difference of mean returns	Relative difference of measure *C*
Π_M1	Pearson	No	0.00011 %	0	0.01523	0
Π_B1	Pearson	Yes	0.01632 %	0.00008	2.29456	0.01441
Π_M2	Spearman	No	0.00007 %	0	0.79151	0
Π_B2	Spearman	Yes	0.02201 %	0.00017	9.41711	0.02952
Π_M3	Kendall	No	0.01245 %	0.00012	6.85256	0.02241
Π_B3	Kendall	Yes	0.03923 %	0.00073	21.59450	0.14168

Source: authors' calculation

Tab. 6: Differences between min-var portfolios and their SSD dominating portfolios, 2008

Portfolio	Correlation *(R)*	Short selling	Absolute difference of mean returns	Absolute difference of measure *C*	Relative difference of mean returns	Relative difference of measure *C*
Π_M1	Pearson	No	0.00119 %	0	0.08500	0
Π_B1	Pearson	Yes	0.00745 %	0.00005	0.85804	0.02125
Π_M2	Spearman	No	0.00041 %	0	0.02776	0
Π_B2	Spearman	Yes	0.00713 %	0.00005	0.74435	0.02109
Π_M3	Kendall	No	0.00584 %	0.00006	0.34901	0.02667
Π_B3	Kendall	Yes	0.02314 %	0.00056	1.42445	0.23594

Source: authors' calculation

In both data sets, the portfolio with the smallest measure of SSD inefficiency (absolute difference of mean returns) is one with minimal Spearman measure of concordance when no short sales are allowed. However, comparing the relative differences of mean returns in 2007 data, one can see that portfolio with minimal Pearson measure of concordance performs better than one with minimal Spearman measure of concordance. Anyway, it is evident that applying Kendall measure of concordance leads to larger SSD inefficiency. Therefore, we suggest using either Pearson or Spearman measure of concordance. We can see that portfolios with minimal measures of concordance in Black model are more SSD inefficient than that in Markowitz model. Moreover, the higher values of SSD inefficiency measure correspond to higher values of differences of concordance measure. Comparing the results for 2007 with that of 2008 we can see that min-var portfolios were less SSD inefficient in 2008 than the year before (except of Π_M1). The absolute differences of concordance measures are smaller in 2008, too.

Conclusions

In this paper we have studied the (in)efficiency of several FX rate portfolios with minimal risk, when the dependency matrix is build up on the basis of alternative concordance measures (namely, Pearson and Kendall measures of dependency). We have defined the efficient portfolio in terms of the second order stochastic dominance and analyzed it on the basis of the Kuosmanen test. Moreover, the analysis was executed for two different time series – FX rate returns of 2007 and 2008.

We have observed that almost all min-var portfolios in 2008 have smaller SSD inefficiency measures than corresponding portfolios during the year before. Hence, during the financial crises min-var portfolios have performed better than before the crises. Moreover, from stochastic dominance point of view, the best concordance measure is Spearman or Pearson one. Finally, the choice of a concordance measure has smaller impact on SSD inefficiency than the choice of short sales restrictions.

All these results can be of great value for portfolio managers in banks and other financial institutions. However, before making a final conclusion about the suitability of particular risk and dependency measures in portfolio theory also other measures of dependency should be examined assuming wider series of data.

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